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Abstract

In this paper, we give a control theoretic approach to the slow self-propelled motion of a rigid body in a viscous fluid. The control of the system is the relative velocity of the fluid with respect to the solid on the boundary of the rigid body (the thrust). Our main results show that, there exists a large class of finite dimensional input spaces for which the system is exactly controllable, i.e., one can find controls steering the rigid body in any final position with a prescribed velocity field. The equations we use are motivated by models of swimming of micro-organisms like cilia. We give a control theoretic interpretation of the swimming mechanism of these organisms, which takes place at very low Reynolds numbers. Our aim is to give a control theoretic interpretation of the swimming mechanism of micro-organisms (like ciliata) which is one of the fascinating problems in fluid mechanics.

1 Introduction and main results.

This paper is aimed to contribute to the understanding of the mechanism of swimming of some microscopic organisms from a control theoretic point of view. As already remarked in Taylor [20], for microscopic organisms the inertia forces, "which are the essential element in self-propulsion of all large living or mechanical bodies, are small compared with forces due to viscosity". The question of understanding the mechanism of swimming of microscopic organisms received a considerable attention from both biologists and specialists in fluid mechanics (see, for instance, [20], Lighthill[13], Childress[5], Galdi[9] and reference therein).

An important example of swimming microscopic organisms is furnished by ciliata (see, for instance, Blake [1] or Brennen [3]). We recall, following Galdi [8], [9], that these organisms can be seen as rigid bodies covered by a great number of hair-like organelles called *cilia* which move in a rather complicated way (see Blake and Otto[2] or Brennen and

Winet[4]). In a commonly accepted model (the *layer model*), the rather complex motion of cilia is replaced by a velocity field on a surface enclosing the layer of cilia (see, for instance, Keller and Wu[11]).

In this work we propose a model of the motion of such micro-organisms consisting in a dynamical system whose state at instant t is

$$Z(t) = \begin{pmatrix} \xi(t) \\ \omega(t) \\ \zeta(t) \\ R(t) \end{pmatrix}, \tag{1.1}$$

where $\xi(t)$ (respectively $\zeta(t)$) denotes the velocity (respectively the position) of the mass center of the rigid body and $\omega(t)$ (respectively $R(t)$) represents the angular velocity vector (respectively the rotation matrix with respect to a reference orientation) of the rigid body at instant t.

The system is controlled by the velocity field induced by the motion of cilia. From the mathematical point of view this control can be seen as the difference of the velocities of the fluid and of the solid on the boundary of the rigid body (the thrust). In order to be more precise, let us denote by $S(t) \subset \mathbb{R}^3$ the open bounded set representing the domain occupied by the moving organisms at instant t. The fact that the solid has a rigid motion implies that there exists an open bounded set $S \subset \mathbb{R}^3$ (which will be used as a reference configuration of the solid) such that

$$S(t) = R(t)S + \zeta(t)$$

for all $t \geq 0$, where $R(t)$ is an orthogonal matrix. We assume that the body is surrounded by a viscous incompressible fluid which occupies the domain $F(t) = \mathbb{R}^3 \setminus \overline{S(t)}$ and we denote by $v(y,t)$ the velocity field of the fluid written in a coordinate attached to the rigid body ($y \in F = \mathbb{R}^3 \setminus \overline{S}$). The input function $u = (u_1, \ldots, u_k) : [0, \infty) \to \mathbb{R}^k$ acts via the boundary condition on ∂S

$$v(y,t) = \xi(t) + \omega(t) \times y + \sum_{i=1}^{k} u_i(t)\psi_i(y) \qquad y \in \partial S, \quad t \geq 0, \tag{1.2}$$

The family of functions $\Psi = \{\psi_1, \ldots, \psi_k\}$ is supposed to be given and contained in one of the following spaces

$$\mathcal{U} = \{\varphi \in C^2(\partial S; \mathbb{R}^3) \;\; \big| \;\; \varphi = 0 \;\; \text{outside} \;\; \Gamma\}, \tag{1.3}$$

$$\mathcal{V} = \{\varphi \in C^2(\partial S; \mathbb{R}^3) \;\; \big| \;\; \varphi \cdot n = 0 \;\; \text{on} \;\; \partial S\}, \tag{1.4}$$

$$\mathcal{W} = \mathcal{U} \cap \mathcal{V}, \tag{1.5}$$

where Γ is an open subset of ∂S and where, for $x \in \partial S$, $n(x)$ denotes the unit vector normal to ∂S and oriented towards the interior of ∂S. This means that we are looking for input functions with values in a finite dimensional vector space and possibly satisfying constraints (like being tangential to ∂S or being supported in a subset Γ of ∂S). We endow $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ with the usual C^2 topology so they become Banach spaces.

In the next section, we introduce a simplified model of self-propelled body based on the above assumption (1.2) and on the balance laws of linear and angular momentum. In this model the state trajectory Z, defined in (1.1), satisfies a first order differential system of the form

$$\dot{Z}(t) = f(Z(t)) + B_\Psi u(t), \qquad t \geq 0. \tag{1.6}$$

Above $f : \mathbb{R}^9 \times SO(3) \to \mathbb{R}^9 \times M_3(\mathbb{R})$ is a smooth function (depending only on S), $B : \mathbb{R}^k \to \mathbb{R}^9 \times M_3(\mathbb{R})$ is the input operator (depending on S and on the choice of the family Ψ), and $SO(3)$ denotes the group of rotations in $\mathbb{R}^3$. The precise definitions of the functions f and B will be given in the next section. For $T \geq 0$, we recall that a system of the form (1.6) is said controllable in time T if, for every $Z_0,\ Z_1 \in \mathbb{R}^9 \times SO(3)$ there exists an input function $u \in L^2(0, T; \mathbb{R}^k)$ such that the solution Z of (1.6) satisfies

$$Z(0) = Z_0, \qquad Z(T) = Z_1.$$

The first main result of this paper is the following.

Theorem 1.1. *Assume that the boundary of S is of class C^2, that Γ is an arbitrary open subset of ∂S and that $k = 6$. Let $\mathcal{Y}_1$ be the subset of those $\Psi = (\psi_1, \ldots, \psi_6) \in \mathcal{U}^6$ such that the system* (1.6) *is controllable in any time $T > 0$ and let $\mathcal{Y}_2$ be the subset of those $\Psi \in \mathcal{V}^6$ such that the system* (1.6) *is controllable in any time $T > 0$. Then $\mathcal{Y}_1$ (respectively $\mathcal{Y}_2$) contains an open dense subset of $\mathcal{U}^6$ (respectively of $\mathcal{V}^6$).*

The above result says that the motion of S can be controlled for a "large" choice of Ψ with velocity fields which are supported in Γ or they are tangential to the boundary. The second main result shows that, by assuming that ∂S is analytic we can control the motion of S with velocity fields which are both tangential to the boundary and vanishing outside Γ.

Theorem 1.2. *Assume that the boundary of S is analytic, that Γ is an arbitrary open subset of ∂S and that $k = 6$. Let $\mathcal{Y}_3$ be the subset of those $\Psi \in \mathcal{W}^6$ such that the system* (1.6) *is controllable in any time $T > 0$. Then $\mathcal{Y}_3$ contains an open dense subset of $\mathcal{W}^6$.*

Moreover, we give several examples of families Ψ which ensure the controllability property.

The plan of this paper is as follows. In Section 2 we describe the mathematical model. In Section 3 we show that our model reduces to a finite dimensional dynamical system. Section 4 contains the proof of our main results. Finally, in Section 5 we give some examples of families Ψ for which the system (1.6) is controllable.

2 The Mathematical Model

The full system modelling the motion of a rigid body into a viscous incompressible fluid is composed of the non stationary Navier-Stokes equations for the fluid coupled to ordinary differential equations (coming from Newton's laws) for the rigid body. More precisely the system is described by the following equations:

$$\rho_F \left(\frac{\partial \overline{v}}{\partial t} + (\overline{v} \cdot \nabla)\overline{v} \right) - \mu \Delta \overline{v} + \nabla \overline{p} = 0, \qquad x \in F(t),\ t \in (0,T), \tag{2.1}$$

$$\operatorname{div} \overline{v} = 0, \qquad x \in F(t),\ t \in (0,T), \tag{2.2}$$

$$\overline{v}(x,t) = \overline{\xi}(t) + \overline{\omega}(t) \times (x - \zeta(t)) + \overline{U}, \qquad x \in \partial S(t),\ t \in [0,T], \tag{2.3}$$

$$\lim_{|y| \to \infty} \overline{v}(y) = 0, \qquad t \in (0,T), \tag{2.4}$$

$$m\dot{\overline{\xi}}(t) = -\int_{\partial S(t)} \sigma(\overline{v}, \overline{p}) n \ \mathrm{d}\Gamma, \qquad t \in (0,T), \tag{2.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\overline{J} \overline{\omega} \right)(t) = -\int_{\partial S(t)} y \times \sigma(\overline{v}, \overline{p}) n \ \mathrm{d}\Gamma, \qquad t \in (0,T), \tag{2.6}$$

$$\dot{\zeta}(t) = \overline{\xi}(t), \qquad t \in (0,T), \tag{2.7}$$

$$\dot{R}(t) = \mathbb{S}(\overline{\omega}(t)) R(t) \qquad t \in (0,T). \tag{2.8}$$

The domains $S(t)$ and $F(t)$ are defined by

$$S(t) = \{x \in \mathbb{R}^3 \ ; \ x = R(t)y + \zeta(t), \quad y \in S\}, \quad F(t) = \mathbb{R}^3 \setminus \overline{S}(t).$$

We can assume, without loss of generality, that the mass center of S is located at the origin. In this case, the unknowns $\zeta(t) \in \mathbb{R}^3$, respectively $R(t) \in SO(3)$, in the above system, stand for the position vector of the mass center, respectively the orientation matrix, of the solid $S(t)$. The other unknowns in the above system are the velocity field of the fluid $\overline{v}$, the pressure field in the fluid $\overline{p}$, the linear velocity of the mass center of the solid $\overline{\xi}$ and the angular velocity of the solid $\overline{\omega}$. Moreover, we have denoted by $\sigma(v,p)$ the stress tensor (also called the Cauchy stress), which is defined by

$$\sigma(v,p) = -pI_3 + 2\mu D(v),$$

where I_3 is the identity matrix of $M_3(\mathbb{R})$ and $D(v)$ is the tensor field defined by

$$D(v)_{k,l} = \frac{1}{2} \left(\frac{\partial v_k}{\partial y_l} + \frac{\partial v_l}{\partial y_k} \right).$$

The positive constant μ is the dynamical viscosity of the fluid. We have denoted by ρ_F the positive density of the fluid. The constant m is the mass of the rigid body whereas $\overline{J}$ denotes the inertia matrix of the rigid body. If we denote by $\rho > 0$ the density of the solid, then we have that

$$m = \int_S \rho \ \mathrm{d}x, \quad \overline{J}_{i,j} = \int_{S(t)} \rho (e_i \times (x - \overline{\zeta})) \cdot (e_i \times (x - \overline{\zeta})) \ \mathrm{d}x.$$

Moreover, for any function w depending only on time, we have denoted by $\dot{w}$ its time derivative.

In (2.8) we have denoted

$$\mathbb{S}(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \quad (\omega \in \mathbb{R}^3). \tag{2.9}$$

It is well-known that the application $\mathbb{S}$ is an isomorphism from $\mathbb{R}^3$ onto the space $A(3)$ of the skew-symmetric matrices and that equation (2.8) could be equivalently written as

$$\dot{R}(t)x = \overline{\omega}(t) \times (R(t)x) \quad (x \in \mathbb{R}^3).$$

Equations (2.1)-(2.8) determine an infinite dimensional nonlinear system. Moreover, since the domain filled by the fluid is not a priori known, we have here a free boundary problem. Therefore the study of (2.1)-(2.8) is a difficult mathematical question. The wellposedness of this system have been extensively studied in recent literature (see, for instance, [15], [17], or [19]). However, questions like controllability or stabilizability of (2.1)-(2.8) are (in the above infinite dimensional setting) open questions. In order to tackle these questions, we derive a simplified finite dimensional model (still nonlinear) aimed to approximate (2.1)-(2.8) in the case of *slow motions* (in a sense which will be made precise later).

Since the equations (2.1)-(2.8) are not written in a cylindrical domain, it is classical (see, for instance, Serre [16]) to use the following change of variables to transform the equations for the fluid into a system written in the fixed domain $F = \mathbb{R}^3 \setminus \overline{S}$:

$$\begin{gathered} y = R^*(t)(x - \zeta(t)), \\ v(y,t) = R^*(t)\overline{v}(\zeta(t) + R(t)y, t), \quad p(y,t) = \overline{p}(\zeta(t) + R(t)y, t), \\ \xi(t) = R^*(t)\overline{\xi}(t), \quad \omega(t) = R^*(t)\overline{\omega}(t). \end{gathered}$$

The above functions satisfy the following problem:

$$\begin{aligned} \rho_F\left(\frac{\partial v}{\partial t} + ([v - \xi - \omega \times y]\cdot\nabla)\, v + \omega \times v\right) - \mu\Delta v + \nabla p = 0, &\qquad y \in F,\ t \in (0,T), \\ \operatorname{div}\, v = 0, &\qquad y \in F,\ t \in (0,T), \\ v(y,t) = \xi(t) + \omega(t) \times y + U, &\qquad y \in \partial S,\ t \in [0,T], \\ \lim_{|y|\to\infty} v(y) = 0, &\qquad t \in (0,T), \\ m\dot{\xi}(t) = -\int_{\partial S} \sigma(v,p)n \, \mathrm{d}\Gamma - m\omega \times \xi, &\qquad t \in (0,T), \\ J\dot{\omega}(t) = (J\omega)\times\omega - \int_{\partial S} y \times \sigma(v,p)n \, \mathrm{d}\Gamma, &\qquad t \in (0,T), \\ \dot{\zeta}(t) = R(t)\xi(t), &\qquad t \in (0,T), \\ \dot{R}(t) = R(t)\mathbb{S}(\omega(t)) &\qquad t \in (0,T). \end{aligned}$$

The above system can written in dimensionless variables. More precisely, following [5], we consider some characteristic length L, some characteristic time τ, and some characteristic speed V. We define the following dimensionless variables

$$\begin{gathered} t^* = \tau^{-1}t, \quad y^* = L^{-1}y, \quad v^* = V^{-1}v, \quad p^* = L(V\mu)^{-1}p, \\ \xi^* = V^{-1}\xi, \quad \omega^* = LV^{-1}\omega, \quad \zeta^* = L^{-1}\zeta, \\ m^* = L^{-3}m, \quad J^* = L^{-5}J, \quad T^* = \tau^{-1}T, \quad U^* = V^{-1}U. \end{gathered}$$

Then the above system can be written as

$$\mathrm{Re}\;\Sigma\frac{\partial v^*}{\partial t^*} + \;\mathrm{Re}\;\left(([v^* - \xi^* - \omega^* \times y^*]\cdot\nabla^*)\, v^* + \omega^* \times v^*\right) - \Delta v^* + \nabla^* p^* = 0,$$
$$y^* \in F^*,\ t^* \in (0,T^*), \quad (2.10)$$

$$\operatorname{div}^* v^* = 0, \quad y^* \in F^*, \ t^* \in (0, T^*), \tag{2.11}$$

$$v^*(y^*, t^*) = \xi^*(t^*) + \omega^*(t^*) \times y^* + U^*, \quad y^* \in \partial S^*, \ t^* \in [0, T^*], \tag{2.12}$$

$$\lim_{|y^*| \to \infty} v^*(y^*) = 0, \quad t^* \in (0, T^*), \tag{2.13}$$

$$\text{Re } \Sigma \frac{m^*}{\rho_F} \dot{\xi}^*(t^*) = -\int_{\partial S^*} \sigma^*(v^*, p^*) n \ \mathrm{d}\Gamma^* - \text{ Re } \Sigma \frac{m^*}{\rho_F} \omega^* \times \xi^*, \quad t^* \in (0, T^*), \tag{2.14}$$

$$\text{Re } \Sigma \frac{J^*}{\rho_F} \dot{\omega}^*(t^*) = \text{ Re } \Sigma \left(\frac{J^*}{\rho_F} \omega^* \right) \times \omega^* - \int_{\partial S^*} y^* \times \sigma^*(v^*, p^*) n \ \mathrm{d}\Gamma^*, \quad t^* \in (0, T^*), \tag{2.15}$$

$$\Sigma \dot{\zeta}^*(t^*) = R(t^*) \xi^*(t^*), \qquad t^* \in (0, T^*), \tag{2.16}$$

$$\dot{R}(t^*) = R(t^*) \mathbb{S}(\omega^*(t^*)), \qquad t^* \in (0, T^*). \tag{2.17}$$

In the above system, we have used the following dimensionless parameters

$$\Sigma = \frac{L}{\tau V} \quad \text{frequency parameter,}$$

$$\text{Re } = \frac{\rho_F V L}{\mu} \quad \text{Reynolds number.}$$

In the case of the swimming of microscopic organisms, the above system can be simplified, by using the fact that the motion of the fluid is a very slow one : the Reynolds number is, in this case, of the order of 10^{-1} and the frequency parameter of order 1 (see, for instance, Childress [5, ch.2]). Therefore, we neglect the first two terms in the right hand side of (2.10) so that the motion of the fluid is modeled by the stationary linear Stokes equations. This means that, although the flow field is time dependent, insofar as the dynamics of the fluid is concerned, it is moving slowly (quasi-steady).

Concerning the solid part, depending on the relative magnitude of the density of the microscopic organisms (with respect to the density of the fluid) the term corresponding to the time derivative and the nonlinear terms in equations (2.14) and (2.15) could be neglected or not. In this paper, we do not make any assumption on the density ρ of the solid and so we will keep all these terms. The mathematical analysis of the models obtained by neglecting these terms is quite similar to the analysis in the next sections so that most of our results would also apply for these models.

For the sake of simplicity, in the remaining part of the paper we omit the exponent $*$, we assume that $\Sigma = 1$ and we use the notation m and J for the parameters

$$m = \text{ Re } \frac{m^*}{\rho_F}, \quad J = \text{ Re } \frac{J^*}{\rho_F}.$$

With the above assumptions the system (2.10)-(2.17) simplifies to

$$-\Delta v + \nabla p = 0, \qquad \text{in } F \times (0,T), \tag{2.18}$$

$$\operatorname{div} v = 0, \qquad \text{in } F \times (0,T), \tag{2.19}$$

$$v(y,t) = \xi(t) + \omega(t) \times y + \sum_{i=1}^{k} u_i(t)\psi_i(y), \qquad y \in \partial S,\ t \in (0,T), \tag{2.20}$$

$$\lim_{|y|\to\infty} v(y) = 0, \qquad t \in (0,T), \tag{2.21}$$

$$m\dot{\xi}(t) = -\int_{\partial S} \sigma(v,p)n \, \mathrm{d}\Gamma + m\omega \times \xi, \qquad t \in (0,T), \tag{2.22}$$

$$J\dot{\omega}(t) = -\int_{\partial S} y \times \sigma(v,p)n \, \mathrm{d}\Gamma + (J\omega) \times \omega, \qquad t \in (0,T), \tag{2.23}$$

$$\dot{\zeta}(t) = R(t)\xi(t), \qquad t \in (0,T), \tag{2.24}$$

$$\dot{R}(t) = R(t)\mathbb{S}(\omega(t)), \qquad t \in (0,T), \tag{2.25}$$

$$\xi(0) = \xi_0,\ \ \omega(0) = \omega_0, \qquad y \in F, \tag{2.26}$$

$$\zeta(0) = \zeta_0,\ \ R(0) = R_0, \qquad y \in F. \tag{2.27}$$

The function $u = (u_1, \ldots, u_k) \in L^2(0,T;\mathbb{R}^k)$ is the control of the system and $\Psi = \{\psi_1, \ldots, \psi_k\}$ is a fixed subset of $C^2\left(\partial S;\mathbb{R}^3\right)$. As shown in Section 3, equations (2.18)-(2.27) determine a nonlinear finite dimensional system. In the remaining part of this paper we study the controllability of this system and we do no longer consider the infinite dimensional system (2.1)-(2.8).

3 Dynamical system formulation

The simplifying assumption of neglecting the term containing the derivative with respect to time in the fluid equation enables us to write (2.18)-(2.27) as a dynamical system in $\mathbb{R}^9 \times SO(3)$. In order to make this assertion precise, we introduce some auxiliary fields (see [10, ch.5] or [8]). Assume that (e_i) is an orthonormal basis of $\mathbb{R}^3$. We define $(h^{(i)},\ p^{(i)})$ (respectively $(H^{(i)},\ P^{(i)})$) as the solution of the following boundary value problem for the Stokes system.

$$\left\{\begin{aligned} -\Delta h^{(i)} + \nabla p^{(i)} &= 0, \quad \text{in } F, \\ \operatorname{div} h^{(i)} &= 0, \quad \text{in } F, \\ h^{(i)}(y) &= e_i, \quad y \in \partial S, \\ \lim_{|y|\to\infty} h^{(i)}(y) &= 0, \end{aligned}\right. \tag{3.1}$$

respectively

$$\left\{\begin{aligned} -\Delta H^{(i)} + \nabla P^{(i)} &= 0, \quad \text{in } F, \\ \operatorname{div} H^{(i)} &= 0, \quad \text{in } F, \\ H^{(i)}(y) &= e_i \times y, \quad y \in \partial S, \\ \lim_{|y|\to\infty} H^{(i)}(y) &= 0. \end{aligned}\right. \tag{3.2}$$

We denote

$$g^{(i)} = \sigma(h^{(i)}, p^{(i)})n|_{\partial F}, \quad i \in \{1,2,3\}, \tag{3.3}$$

$$G^{(i)} = \sigma(H^{(i)}, P^{(i)})n|_{\partial F}, \quad i \in \{1,2,3\}. \tag{3.4}$$

For homogeneous Sobolev spaces we use, following [7], the notation

$$D^{l,q}(F) = \left\{ u \in L^1_{\mathrm{loc}}(F) \;\middle|\; \partial^\alpha u \in L^q(F) \;\text{ for all }\; \alpha \in \mathbb{N}^3,\; |\alpha| = l \right\},$$

with $l \in \mathbb{N}$, $1 \leq q \leq \infty$.

The following result shows that the above systems are well-posed.

Lemma 3.1. *1. Assume that the boundary ∂S is of class C^2. Then the systems* (3.1) *and* (3.2) *admit unique solutions such that*

$$h^{(i)},\; H^{(i)} \in L^s(F) \cap D^{1,r}(F) \cap D^{2,q}(F) \cap C^\infty(F),$$

$$p^{(i)},\; P^{(i)} \in L^r(F) \cap D^{1,q}(F) \cap C^\infty(F)$$

for $s \in (3, \infty]$, $r \in (\frac{3}{2}, \infty]$, $q \in (1, \infty)$. Moreover, we have that

$$\|(1 + |y|)h^{(i)}\|_{L^\infty(F)} < \infty,$$

$$\|(1 + |y|)H^{(i)}\|_{L^\infty(F)} < \infty.$$

2. Assume that the boundary ∂S is analytic. Then $(h^{(i)}, p^{(i)})$ and $(H^{(i)}, P^{(i)})$ are analytic up to the boundary.

Proof. The first result comes from the classical wellposedness results for the Stokes system (see, for instance, [7, Chapter V], [17]). For the analyticity we refer to Komatsu [12] and Morrey [14]. □

We next introduce several matrices playing an important role in the remaining part of the paper. For $i, j \in \{1, 2, 3\}$ we denote

$$K_{i,j} = -\int_{\partial S} g_j^{(i)} \,\mathrm{d}\Gamma, \qquad C_{i,j} = -\int_{\partial S} \left(x \times g^{(i)}\right)_j \,\mathrm{d}\Gamma, \tag{3.5}$$

$$\widetilde{C}_{i,j} = -\int_{\partial S} G_j^{(i)} \,\mathrm{d}\Gamma, \qquad \Omega_{i,j} = -\int_{\partial S} \left(x \times G^{(i)}\right)_j \,\mathrm{d}\Gamma. \tag{3.6}$$

It is known (see, for instance, [10, ch.5]) that $\widetilde{C} = C^*$ and that the matrix $A \in M_6(\mathbb{R})$ defined by

$$A = \begin{pmatrix} m^{-1}K & m^{-1}C \\ J^{-1}C^* & J^{-1}\Omega \end{pmatrix}, \tag{3.7}$$

is self-adjoint and negative-definite, provided that we endow $\mathbb{R}^6$ with the inner product

$$\langle a, b \rangle = m \sum_{q=1}^{3} a_q b_q + \sum_{p,q=1}^{3} J_{p,q} a_{q+3} b_{p+3}\,.$$

We next introduce the matrices $B^{(1)},\ B^{(2)} \in M_{3\times k}(\mathbb{R})$ and $B \in M_{6\times k}$ defined by

$$B^{(1)}_{i,j} = -\int_{\partial S} g^{(i)} \cdot \psi_j \,\mathrm{d}\Gamma, \quad i \in \{1, 2, 3\}, \quad j \in \{1, \ldots, k\}, \tag{3.8}$$

$$B^{(2)}_{i,j} = -\int_{\partial S} G^{(i)} \cdot \psi_j \,\mathrm{d}\Gamma, \quad i \in \{1, 2, 3\}, \quad j \in \{1, \ldots, k\}, \tag{3.9}$$

$$B = \begin{pmatrix} m^{-1}B^{(1)} \\ J^{-1}B^{(2)} \end{pmatrix}. \tag{3.10}$$

For given input functions $u_1, \dots, u_k \in L^2(0,T;\mathbb{R})$, according to [7, Chapter V], there exists a unique solution (U, Q) of

$$-\Delta U + \nabla Q = 0, \qquad \text{in } F \times (0,T), \tag{3.11}$$

$$\operatorname{div} U = 0, \qquad \text{in } F \times (0,T), \tag{3.12}$$

$$U(y,t) = \sum_{i=1}^{k} u_i(t)\psi_i(y), \quad y \in \partial S,\ t \in (0,T), \tag{3.13}$$

$$\lim_{|y|\to\infty} U(y) = 0, \qquad t \in (0,T) \tag{3.14}$$

satisfying

$$U(\cdot,t) \in L^s(F) \cap D^{1,r}(F) \cap D^{2,q}(F) \cap C^\infty(F),$$
$$Q(\cdot,t) \in L^r(F) \cap D^{1,q}(F) \cap C^\infty(F)$$

for $s \in (3,\infty]$, $r \in (\frac{3}{2},\infty]$, $q \in (1,\infty)$ and

$$\operatorname*{ess\,sup}_{y\in F}\ (1+|y|)\,|U(y,t)| < \infty,$$

for almost every $t \in (0,T)$.

For a, $b \in \mathbb{R}^3$, we also set

$$E\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} b \times a \\ J^{-1}\left((Jb)\times b\right) \end{pmatrix}. \tag{3.15}$$

We are now in a position to prove that equations (2.18)-(2.27) determine a finite dimensional dynamical system.

Lemma 3.2. *Assume that $T > 0$ and that $u \in L^2(0,T;\mathbb{R}^m)$. Then $(v, p, \xi, \omega, \zeta, R)$ satisfy* (2.18)-(2.27) *together with*

$$v \in H^1(0,T; L^s(F) \cap D^{1,r}(F) \cap D^{2,q}(F) \cap C^\infty(F)), \tag{3.16}$$

$$p \in H^1(0,T; L^r(F) \cap D^{1,q}(F) \cap C^\infty(F)), \tag{3.17}$$

$$\xi \in H^1(0,T;\mathbb{R}^3),\ \omega \in H^1(0,T;\mathbb{R}^3),\ \zeta \in C^1([0,T];\mathbb{R}^3),\ R \in C^1([0,T];SO(3)), \tag{3.18}$$

for $s \in (3,\infty]$, $r \in (\frac{3}{2},\infty]$, $t \in (1,\infty)$ if and only if

$$\begin{pmatrix} \dot\xi(t) \\ \dot\omega(t) \end{pmatrix} = A\begin{pmatrix} \xi(t) \\ \omega(t) \end{pmatrix} + E\left(\begin{pmatrix} \xi(t) \\ \omega(t) \end{pmatrix}\right) + Bu(t), \tag{3.19}$$

$$\dot\zeta(t) = R(t)\xi(t), \qquad \dot R(t) = R(t)\mathbb{S}(\omega(t)), \tag{3.20}$$

$$\xi(0) = \xi_0,\ \ \omega(0) = \omega_0,\ \ \zeta(0) = \zeta_0,\ \ R(0) = R_0, \tag{3.21}$$

$$v = \sum_i \xi_i h^{(i)} + \omega_i H^{(i)} + U, \quad p = \sum_i \xi_i p^{(i)} + \omega_i P^{(i)} + Q, \tag{3.22}$$

where A, B are the matrices defined by (3.7),(3.10), *$E(\cdot)$ is defined by* (3.15) *and (U, Q) is the solution of* (3.11)-(3.14).

Proof. Assume that $(v, p, \xi, \omega, \zeta, R)$ satisfies (3.19)-(3.22). From (3.1), (3.2) and (3.11)-(3.14), it is easy to check that (v, p) satisfies (2.18)-(2.20). From (3.19) and from the definitions of A and B, it follows that for $k \in \{1, 2, 3\}$ we have

$$m\dot{\xi}_k = -\int_{\partial F} \sum_{i=1}^{3} \left(\xi_i g_k^{(i)} + \omega_i G_k^{(i)}\right) \, \mathrm{d}\Gamma - \int_{\partial S} \sum_{j=1}^{3} u_j \psi_j \cdot g^{(k)} \, \mathrm{d}\Gamma + m\,(\omega \times \xi)_k\,, \tag{3.23}$$

$$(J\dot{\omega})_k = -\int_{\partial F} \sum_{i=1}^{3} \left[\xi_i (y \times g^{(i)})_k + \omega_i (y \times G^{(i)})_k\right] \, \mathrm{d}\Gamma - \int_{\partial S} \sum_{j=1}^{3} u_j \psi_j \cdot G^{(k)} \, \mathrm{d}\Gamma + \left[(J\omega) \times \omega\right]_k . \tag{3.24}$$

By using (3.3) and (3.13), the last term in the right-hand side of (3.23) can be written as

$$\int_{\partial S} \sum_{j=1}^{3} u_j \psi_j \cdot g^{(k)} \, \mathrm{d}\Gamma = \int_{\partial S} \sigma(h^{(k)}, p^{(k)}) n \cdot U \, \mathrm{d}\Gamma.$$

The above relation, combined to (3.30), implies that

$$\int_{\partial S} \sum_{j=1}^{3} u_j \psi_j \cdot g^{(k)} \, \mathrm{d}\Gamma = \left[\int_{\partial S} \sigma(U, Q) n \, \mathrm{d}\Gamma\right]_k . \tag{3.25}$$

Relations (3.23) and (3.25) clearly imply that

$$m\dot{\xi} = -\int_{\partial F} \sum_{i=1}^{3} \left(\xi_i g^{(i)} + \omega_i G^{(i)}\right) \, \mathrm{d}\Gamma - \int_{\partial F} \sigma(U, Q) n \, \mathrm{d}\Gamma + m\omega \times \xi. \tag{3.26}$$

Similar calculations show that

$$J\dot{\omega} = -\int_{\partial F} \sum_{i=1}^{3} \left[\xi_i (y \times g^{(i)}) + \omega_i (y \times G^{(i)})\right] \, \mathrm{d}\Gamma - \int_{\partial S} y \times \sigma(U, Q) n \, \mathrm{d}\Gamma + (J\omega) \times \omega. \tag{3.27}$$

Relations (3.26) and (3.27) imply that (ξ, ω) satisfies (2.22)-(2.23). Then, by taking in consideration the fact that (ζ, R) satisfy (2.24)-(2.27), we obtain that $(v, p, \xi, \omega, \zeta, R)$ satisfies (2.18)-(2.27).

The proof of the fact that any solution of (2.18)-(2.27) satisfies (3.19)-(3.22) is similar and therefore we don't give it here. □

From the above lemma it easily follows that the following result holds.

Corollary 3.3. *Assume that* $T > 0$ *and that* $u \in L^2(0, T; \mathbb{R}^k)$. *Then there exists a unique solution* $(v, p, \xi, \omega, \zeta, R)$ *of the system* (2.18)-(2.27) *satisfying* (3.16)-(3.18). *More precisely, if we denote* $z(t) = \begin{pmatrix} \xi(t) \\ \omega(t) \end{pmatrix}$ *then equations* (2.18)-(2.27) *determine a dynamical system with input space* $\mathbb{R}^k$, *state space* $\mathbb{R}^9 \times SO(3)$ *and with the state equations*

$$\dot{z}(t) = Az(t) + E(z(t)) + Bu(t), \tag{3.28}$$

$$\dot{\zeta}(t) = R(t)\xi(t), \qquad \dot{R}(t) = R(t)\mathbb{S}(\omega(t)) \tag{3.29}$$

Proof. The local in time existence follows from the Cauchy-Lipschitz Theorem. To get the global in time existence it suffices to show that the solutions do not blow up in finite time. It is easy to check that $\langle E(z), z\rangle = 0$ for every $z \in \mathbb{R}^6$. Therefore, by taking the inner product of (3.28) with $z(t)$ we have that

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 =< Az(t), z(t) > + < Bu(t), z(t) > .$$

The above relation and the fact that A is symmetric and negative-definite imply that

$$\|z(t)\|^2 \leq \|z(0)\|^2 + \int_0^t \left(\|Bu(s)\|^2 + \|z(s)\|^2\right)\, \mathrm{d}s.$$

By the Gronwall lemma it follows that z does not blow up in finite time so that we have proved the global existence result. □

We will also need the following technical lemma.

Lemma 3.4. *With the above notation, for $i \in \{1, 2, 3\}$ and for almost every $t \in (0, T)$, we have that*

$$\int_{\partial S} \sigma(h^{(i)}, p^{(i)})n \cdot U \,\mathrm{d}\Gamma = \left[\int_{\partial S} \sigma(U, Q)n \,\mathrm{d}\Gamma\right]_i, \tag{3.30}$$

$$\int_{\partial S} \sigma(H^{(i)}, P^{(i)})n \cdot U \,\mathrm{d}\Gamma = \left[\int_{\partial S} x \times \sigma(U, Q)n \,\mathrm{d}\Gamma\right]_i. \tag{3.31}$$

Proof. By taking the inner product of (3.11) with $h^{(i)}$ we obtain that

$$\int_F \operatorname{div} \sigma(U, Q) \cdot h^{(i)} \,\mathrm{d}y = 0.$$

Using an integration by parts, the above relation implies that

$$\int_{\partial S} \sigma(U, Q)n \cdot h^{(i)} \mathrm{d}\,\Gamma = \int_F D(U) : D(h^{(i)}) \,\mathrm{d}y. \tag{3.32}$$

Similarly, by taking the inner product of (3.1) with U we obtain that

$$\int_F \operatorname{div} \sigma(h^{(i)}, p^{(i)}) \cdot U \,\mathrm{d}y = 0.$$

Integrating by parts we get that

$$\int_{\partial S} \sigma(h^{(i)}, p^{(i)})n \cdot U \mathrm{d}\,\Gamma = \int_F D(U) : D(h^{(i)}) \,\mathrm{d}y. \tag{3.33}$$

From (3.32) and (3.33) and the fact that $h^{(i)} = e_i$ on ∂S, we conclude that (3.30) holds true.

The proof of (3.31) is similar so we skip it here. □

4 Proof of the main results

The main ingredients of the proof of Theorem 1.1 are the two following lemmas.

Lemma 4.1. *Assume that Γ is a non empty open subset of ∂S (with respect to the induced topology).*

1. *Suppose that ∂S is of class C^2. Then the family*
$$\mathcal{F}_1 = \left\{ g^{(1)}, g^{(2)}, g^{(3)}, G^{(1)}, G^{(2)}, G^{(3)} \right\}$$
is linearly independent in $L^2(\Gamma, \mathbb{R}^3)$ and the family
$$\mathcal{F}_2 = \left\{ \left(g^{(i)} \times n\right) \times n, \left(G^{(i)} \times n\right) \times n \;\middle|\; i \in \{1,2,3\} \right\}$$
is linearly independent in $L^2(\partial S, \mathbb{R}^3)$.

2. *Suppose that ∂S is analytic. Then $\mathcal{F}_2$ is linearly independent in $L^2(\Gamma, \mathbb{R}^3)$.*

Proof. 1. Let us consider $\gamma, \delta \in \mathbb{R}^3$ such that
$$\sum_{1 \leq i \leq 3} \left(\gamma_i g^{(i)} + \delta_i G^{(i)} \right) = 0 \quad \text{on } \Gamma. \tag{4.1}$$

Then, we denote
$$H = \sum_{1 \leq i \leq 3} \left(\gamma_i h^{(i)} + \delta_i H^{(i)} \right), \quad P = \sum_{1 \leq i \leq 3} \left(\gamma_i p^{(i)} + \delta_i P^{(i)} \right). \tag{4.2}$$

From (4.1) it follows that
$$\sigma(H, P) n = 0 \quad \text{on } \Gamma. \tag{4.3}$$

Moreover, from the definition (3.1) and (3.2) of $(h^{(i)}, p^{(i)})$ and $(H^{(i)}, P^{(i)})$ we have that (H, P) satisfies
$$\begin{cases} -\Delta H + \nabla P = 0, & \text{in } F, \\ \operatorname{div} H = 0, & \text{in } F, \\ H = \gamma + \delta \times y, & y \in \partial S, \\ \lim\limits_{|y| \to \infty} H(y) = 0. \end{cases} \tag{4.4}$$

We consider $\mathcal{O} \subset S$ an open set such that
$$\overline{\mathcal{O}} \cap \overline{F} \subset \Gamma.$$

We then define $\widetilde{H}$ and $\widetilde{P}$ on $F \cup \mathcal{O}$ by the following formulas: $\widetilde{H} = H - (\gamma + \delta \times y)$ for all $y \in F$ and $\widetilde{H} = 0$ for all $y \in \mathcal{O}$ and $\widetilde{P} = P$ for all $y \in F$ and $\widetilde{P} = 0$ for all $y \in \mathcal{O}$. By using (4.3) it follows that $(\widetilde{H}, P) \in H^1_{loc}(F \cup \mathcal{O}) \times L^2_{loc}(F \cup \mathcal{O})$ and
$$\begin{aligned} -\Delta \widetilde{H} + \nabla \widetilde{P} = 0, &\quad \text{in } F \cup \mathcal{O}, \\ \operatorname{div} \widetilde{H} = 0, &\quad \text{in } F \cup \mathcal{O}, \\ \widetilde{H} = 0, &\quad \text{in } \mathcal{O}. \end{aligned}$$

By using the unique continuation result of Fabre and Lebeau [6, Theorem 1], we deduce that $\widetilde{H} = 0$, so that $H = \gamma + \delta \times y$ for all $y \in F$. Since $\lim_{|y| \to \infty} H(y) = 0$, we have that $\gamma = \delta = 0$. Consequently, the family $\mathcal{F}_1$ is linearly independent.

For the linear independence of $\mathcal{F}_2$ in $L^2(\partial S, \mathbb{R}^3)$ we refer to Lemma 2.1 of [8].

2. Let us consider $\gamma, \delta \in \mathbb{R}^3$ such that

$$\sum_{1\le i\le 3} \Big(\gamma_i (g^{(i)} \times n) \times n + \delta_i (G^{(i)} \times n) \times n\Big) = 0 \quad \text{on } \Gamma. \tag{4.5}$$

Then, since the function

$$x \mapsto \sum_{i=1}^{3} \Big[\gamma_i (g^{(i)} \times n) \times n + \delta_i (G^{(i)} \times n) \times n\Big],$$

is analytic on ∂S, it follows that

$$\sum_{i=1}^{3} \Big[\gamma_i (g^{(i)} \times n) \times n + \delta_i (G^{(i)} \times n) \times n\Big] = 0, \quad \text{on } \partial S.$$

The linear independence of $\mathcal{F}_2$ in $L^2(\Gamma, \mathbb{R}^3)$ follows now from the fact that $\mathcal{F}_2$ is linearly independent in $L^2(\partial S, \mathbb{R}^3)$.

□

The result below shows that for $k = 6$ the mapping associating the matrix B to the family $\{\psi_1, \dots \psi_6\}$ via formulas (3.8)-(3.10) is, under quite general assumptions, onto.

Lemma 4.2. *Assume that* Γ *is a non empty open subset of* ∂S *(with respect to the induced topology). Then the linear operator* $\Lambda : [L^2(\partial S; \mathbb{R}^3)]^6 \to M_{6\times 6}(\mathbb{R})$*, defined by*

$$\Lambda \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_6 \end{pmatrix} = - \begin{pmatrix} \left(\displaystyle\int_{\partial S} g^{(i)} \cdot \psi_j \, \mathrm{d}\Gamma\right)_{i\in\{1,2,3\}, j\in\{1,\dots,6\}} \\ \left(\displaystyle\int_{\partial S} G^{(i)} \cdot \psi_j \, \mathrm{d}\Gamma\right)_{i\in\{1,2,3\}, j\in\{1,\dots,6\}} \end{pmatrix} \tag{4.6}$$

is such that

1. $\Lambda\left(\mathcal{U}^6\right) = \Lambda\left(\mathcal{V}^6\right) = M_{6\times 6}(\mathbb{R})$
2. *if* ∂S *is analytic, then* $\Lambda\left(\mathcal{W}^6\right) = M_{6\times 6}(\mathbb{R})$

Proof. We first set

$$\begin{gathered} \mathbb{U} = \{(\varphi_i) \in L^2(\partial S; \mathbb{R}^3)^6 \;\mid\; \varphi_i = 0 \;\text{ outside }\; \Gamma,\; i \in \{1, \dots, 6\}\}, \\ \mathbb{V} = \{(\varphi_i) \in L^2(\partial S; \mathbb{R}^3)^6 \;\mid\; \varphi_i \cdot n = 0 \;\text{ on }\; \partial S,\; i \in \{1, \dots, 6\}\}, \\ \mathbb{W} = \mathbb{U} \cap \mathbb{V}, \end{gathered}$$

and we denote by $\Lambda_{\mathbb{U}}$ (respectively $\Lambda_{\mathbb{V}}$, $\Lambda_{\mathbb{W}}$) the restriction of Λ to $\mathbb{U}$ (respectively $\mathbb{V}$, $\mathbb{W}$). Simple calculations show that for all $C \in M_6(\mathbb{R})$ we have

$$\Lambda_{\mathbb{U}}^*(C) = \begin{pmatrix} \displaystyle\sum_{k=1}^{3} \Big[C_{k,1} g^{(k)} + C_{k+3,1} G^{(k)}\Big] \\ \vdots \\ \displaystyle\sum_{k=1}^{3} \Big[C_{k,6} g^{(k)} + C_{k+3,6} G^{(k)}\Big] \end{pmatrix} \quad \text{on } \Gamma,$$

$$\Lambda^*_{\mathbb{V}}(C) = \begin{pmatrix} \displaystyle\sum_{k=1}^{3} \left[C_{k,1}(g^{(k)} \times n) \times n + C_{k+3,1}(G^{(k)} \times n) \times n \right] \\ \vdots \\ \displaystyle\sum_{k=1}^{3} \left[C_{k,6}(g^{(k)} \times n) \times n + C_{k+3,6}(G^{(k)} \times n) \times n \right] \end{pmatrix} \quad \text{on } \partial S,$$

$$\Lambda^*_{\mathbb{W}}(C) = \begin{pmatrix} \displaystyle\sum_{k=1}^{3} \left[C_{k,1}(g^{(k)} \times n) \times n + C_{k+3,1}(G^{(k)} \times n) \times n \right] \\ \vdots \\ \displaystyle\sum_{k=1}^{3} \left[C_{k,6}(g^{(k)} \times n) \times n + C_{k+3,6}(G^{(k)} \times n) \times n \right] \end{pmatrix} \quad \text{on } \Gamma.$$

We first prove that the restriction of Λ to $\mathcal{U}^6$ is onto. By using Lemma 4.1, it follows that $\Lambda^*_{\mathbb{U}}$ is one to one. Consequently, the range of $\Lambda_{\mathbb{U}}$ is a dense subspace of $M_6(\mathbb{R})$. By using the density of $\mathcal{U}^6$ in $\mathbb{U}$ we get that $\Lambda\left(\mathcal{U}^6\right)$ is a dense subspace of $M_6(\mathbb{R})$. Since $M_6(\mathbb{R})$ is finite dimensional, all its subspaces are closed so that $\Lambda\left(\mathcal{U}^6\right) = M_6(\mathbb{R})$.

Using again Lemma 4.1, we have that $\Lambda^*_{\mathbb{V}}$ is one to one and if ∂S is analytic that $\Lambda^*_{\mathbb{W}}$ is one to one. Acting as for $\mathcal{U}$, we easily conclude that $\Lambda\left(\mathcal{V}^6\right) = M_6(\mathbb{R})$ and that for ∂S analytic we have that $\Lambda\left(\mathcal{W}^6\right) = M_6(\mathbb{R})$. □

We are now in a position to prove the main results.

Proof of Theorem 1.1. We only prove Theorem 1.1 for the set $\mathcal{Y}_1$. The proof for the set $\mathcal{Y}_2$ is completely similar and it is omitted.

Let us define $\widetilde{\mathcal{Y}}_1$ the set of those Ψ for which the matrix B given by (3.8)-(3.9) is invertible. We first check that $\mathcal{Y}_1$ contains $\widetilde{\mathcal{Y}}_1$ and then that $\widetilde{\mathcal{Y}}_1$ is an open dense subset of $\mathcal{U}^6$. Indeed, assume that

$$Z_0 = \begin{pmatrix} \xi_0 \\ \omega_0 \\ \zeta_0 \\ R_0 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} \xi_1 \\ \omega_1 \\ \zeta_1 \\ R_1 \end{pmatrix} \in \mathbb{R}^9 \times SO(3).$$

It is easy to check that there exist two C^2 functions $\widetilde{\zeta} : [0,T] \to \mathbb{R}^3$, $\widetilde{R} : [0,T] \to SO(3)$ such that

$$\widetilde{\xi}(0) = \xi_0,\ \widetilde{\omega}(0) = \omega_0,\ \widetilde{\zeta}(0) = \zeta_0,\ \widetilde{R}(0) = R_0,$$
$$\widetilde{\xi}(T) = \xi_1,\ \widetilde{\omega}(T) = \omega_1,\ \widetilde{\zeta}(T) = \zeta_1,\ \widetilde{R}(T) = R_1.$$

If we set

$$u(t) = B^{-1}\left(\dot{\widetilde{z}} - A\widetilde{z} - E(\widetilde{z})\right),$$

with $\widetilde{z}(t) = \begin{pmatrix} \widetilde{R}^* \dot{\widetilde{\zeta}} \\ \mathbb{S}^{-1}\left(\widetilde{R}^* \dot{\widetilde{R}}\right) \end{pmatrix}$ then we clearly have $Z(0) = Z_0$, $Z(T) = Z_1$, thus the system is controllable in time T. We have shown that $\mathcal{Y}_1$ contains $\widetilde{\mathcal{Y}}_1$.

On the other hand, by comparing (3.8)-(3.9) and (4.6), we see that $B = \Lambda(\Psi)$. Since the mapping Λ is continuous from $\mathcal{U}^6$ to $M_6(\mathbb{R})$ and since the set of invertible matrices is open in $M_6(\mathbb{R})$, we have that $\widetilde{\mathcal{Y}}_1$ is open in $\mathcal{U}^6$. We next check the density of $\widetilde{\mathcal{Y}}_1$. By Lemma 4.2, there exists $\widetilde{\Psi} \in \mathcal{U}^6$ such that $\Lambda(\widetilde{\Psi}) = I_6$. For $\Psi \in \mathcal{U}^6$, we consider the sequence $\left(\Psi - \frac{1}{j}\widetilde{\Psi}\right)_{j\in\mathbb{N}^*}$ which converges to Ψ. It is easy to check that, excepting a finite number of values of j the matrix $\Lambda\left(\Psi - \frac{1}{j}\widetilde{\Psi}\right)$ is invertible. We have thus shown that $\mathcal{Y}_1$ contains the set $\widetilde{\mathcal{Y}}_1$ which is open and dense in $\mathcal{U}^6$. □

Proof of Theorem 1.2. As in the above proof, we first introduce the set $\widetilde{\mathcal{Y}}_3$ of those Ψ for which B given by (3.8) and (3.9) is invertible.

We can check that $\mathcal{Y}_3$ contains $\widetilde{\mathcal{Y}}_3$ as in the proof of Theorem 1.1. Moreover, using the fact that $B = \Lambda(\Psi)$, that the mapping Λ is continuous from $\mathcal{W}^6$ to $M_6(\mathbb{R})$ and that the set of invertible matrices is open in $M_6(\mathbb{R})$, we get that $\widetilde{\mathcal{Y}}_3$ is open in $\mathcal{W}^6$. Finally, we verify the density of $\widetilde{\mathcal{Y}}_3$. By using again Lemma 4.2, we obtain the existence of $\widetilde{\Psi} \in \mathcal{W}^6$ such that $\Lambda(\widetilde{\Psi}) = I_6$. Then, acting as in the above proof, we show that for all $\Psi \in \mathcal{W}^6$, there exists a sequence $\Psi_n \in \widetilde{\mathcal{Y}}_3$ such Ψ_n converges toward Ψ. □

5 Examples

In this section, we give some examples of families Ψ of vector fields, defined on ∂S, for which the system (2.18)-(2.27) is controllable. Denote

- $\mathcal{F}_0 = \{e_1, e_2, e_3, e_1 \times y, e_2 \times y, e_3 \times y\}$;
- $\mathcal{F}_1 = \left\{g^{(1)}, g^{(2)}, g^{(3)}, G^{(1)}, G^{(2)}, G^{(3)}\right\}$;
- $\mathcal{F}_2 = \left\{ \left(g^{(i)} \times n\right) \times n, \left(G^{(i)} \times n\right) \times n \;\middle|\; i \in \{1,2,3\}\right\}$.

We remark that the families $\mathcal{F}_1$ and $\mathcal{F}_2$ have already been used in the previous sections.

The main result of this section is

Proposition 5.1. *If Ψ is one of the above families, then the system* (2.18)-(2.27) *is exactly controllable in any time $T > 0$.*

Proof. We first remark that it suffices to show that, for $\Psi \in \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ the matrix B defined by (3.8)-(3.10) is invertible.

If $\Psi = \mathcal{F}_0$ then $B = A$ so that B is clearly invertible.

If $\Psi = \mathcal{F}_1$ then

$$B = \begin{pmatrix} mI_3 & 0 \\ 0 & J \end{pmatrix}^{-1} \begin{pmatrix} M & N \\ N^* & O \end{pmatrix}, \tag{5.1}$$

with

$$M_{i,j} = -\int_{\partial S} g^{(i)} \cdot g^{(j)} \, \mathrm{d}\Gamma,\ N_{i,j} = -\int_{\partial S} g^{(i)} \cdot G^{(j)} \, \mathrm{d}\Gamma,\ O_{i,j} = -\int_{\partial S} G^{(i)} \cdot G^{(j)} \, \mathrm{d}\Gamma.$$

According to Lemma 4.1, the system of vector functions $\left\{g^{(i)}, G^{(i)} \mid i \in \{1,2,3\}\right\}$ is linearly independent in $L^2(\partial S; \mathbb{R}^3)$, so that the second matrix in the right-hand side of (5.1) is invertible. Consequently B is invertible in the case $\Psi = \mathcal{F}_1$.

Assume then that $\Psi = \mathcal{F}_2$. Since, in this case, the vector fields belonging to Ψ are tangential, formulas (3.8)-(3.10) yield

$$B = \begin{pmatrix} mI_3 & 0 \\ 0 & J \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{M} & \widetilde{N} \\ \widetilde{N}^* & \widetilde{O} \end{pmatrix}, \tag{5.2}$$

with

$$\begin{aligned}
\widetilde{M}_{i,j} &= \int_{\partial S} \left[\left(g^{(i)} \times n\right) \times n\right] \cdot \left[\left(g^{(j)} \times n\right) \times n\right] \, \mathrm{d}\Gamma, \\
\widetilde{N}_{i,j} &= \int_{\partial S} \left[\left(g^{(i)} \times n\right) \times n\right] \cdot \left[\left(G^{(j)} \times n\right) \times n\right] \, \mathrm{d}\Gamma, \\
\widetilde{O}_{i,j} &= \int_{\partial S} \left[\left(G^{(i)} \times n\right) \times n\right] \cdot \left[\left(G^{(j)} \times n\right) \times n\right] \, \mathrm{d}\Gamma.
\end{aligned}$$

By using again Lemma 2.1 from [8] the system of vector functions

$$\left\{ \left[\left(g^{(i)} \times n\right) \times n\right], \left[\left(G^{(i)} \times n\right) \times n\right] \;\middle|\; i = 1,3 \right\}$$

is linearly independent in $L^2(\partial S; \mathbb{R}^3)$, so that the second matrix in the right-hand side of (5.2) is invertible. Consequently B is invertible for $\Psi = \mathcal{F}_2$. $\square$

The fact that the above choice of the families $\mathcal{F}_1$ and $\mathcal{F}_2$ is physically relevant may be motivated by the following result:

Proposition 5.2. *Assume that $\Psi = \{\psi_i \mid i = 1, \ldots, k\}$ is a family of $C^2(\partial S; \mathbb{R}^3)$. Then we have that*

1. *if $\Psi \subset \mathcal{F}_1^{\perp}$ (the orthogonal is taken in $L^2(\partial S; (\mathbb{R}^3))^6$), then the system* (3.28), (3.29) *is not controllable;*
2. *if $\Psi \subset \mathcal{V} \cap \mathcal{F}_2^{\perp}$ then the system* (3.28), (3.29) *is not controllable.*

More precisely, in the above cases, the control u does not act on the system (3.28), (3.29).

Proof. From (3.8)-(3.10) it follows that $B = 0$ thus, in this case, the input function has no influence on the state of the system. $\square$

An interesting question is to know if the motion can be controlled by using less than six scalar inputs. This question is open in the general case. A partial answer is given by the two results below in the particular case where the rigid body is the unit ball.

We first show that by suppressing an appropriate element of one of the families $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ (the families introduced at the beginning of this section) the resulting system is no longer controllable.

Proposition 5.3. *Assume that S is the unit ball in $\mathbb{R}^3$. Then, for every $i \in \{0,1,2\}$ there exists a set containing five elements $\Psi_i \subset \mathcal{F}_i$ such that the system* (3.28), (3.29) *is not controllable.*

Proof. With the assumption that S is the unit ball in $\mathbb{R}^3$ the fields $g^{(i)}$, $G^{(i)}$ are explicitly given (see, for instance [10, p.163,169]) by the formulas

$$g^{(i)}(y) = \frac{3}{2}e_i, \qquad G^{(i)}(y) = 3e_i \times y, \qquad i \in \{1, 2, 3\}. \tag{5.3}$$

In the next calculation we use the quantities ϵ_{ijk} which are the components of the classical permutation tensor, i.e., the quantities ϵ_{ijk} are skew-symmetric with respect to any to any couple of indexes and $\epsilon_{123} = 1$.

We next inject the expressions (5.3) in the formulas (3.5) and (3.6) which define the matrices K, C, S and Ω we obtain

$$K_{i,j} = -\frac{3}{2}\int_{\partial S} \delta_{ij} \,\mathrm{d}\Gamma = -6\pi\delta_{ij} \qquad \forall i, j \in \{1, 2, 3\},$$

$$C_{i,j} = -\frac{3}{2}\int_{\partial S} \epsilon_{jkl}y_k\delta_{il} \,\mathrm{d}\Gamma = 0 \qquad \forall i, j \in \{1, 2, 3\},$$

$$\begin{aligned}
\Omega_{i,j} &= -3\left[\int_{\partial S} y \times (e_i \times y)\right]_j \\
&= -3\int_{\partial\Omega} \epsilon_{jpq}y_p\epsilon_{qrs}\delta_{ir}y_s \\
&= -4\pi\epsilon_{jpq}\epsilon_{qrs}\delta_{ir}\delta_{ps} \\
&= -4\pi\epsilon_{jpq}\epsilon_{rpq}\delta_{ir} \\
&= -8\pi\delta_{jr}\delta_{ir} \\
&= -8\pi\delta_{ij}.
\end{aligned}$$

Consequently we have

$$K = -6\pi I_3, \quad C = 0, \quad \Omega = -8\pi I_3.$$

The above relations, combined to (3.7), imply that

$$A = \begin{pmatrix} -\frac{3}{2\rho}I_3 & 0 \\ 0 & -\frac{15}{\rho}I_3 \end{pmatrix}. \tag{5.4}$$

We next take $\Psi = \{e_1, e_2, e_3, e_1 \times y, e_2 \times y\}$ which is a five elements subset of $\mathcal{F}_0$. With the above choice of Ψ and by using (3.8)-(3.10) we obtain

$$B = \begin{pmatrix} -\frac{3}{2\rho} & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2\rho} & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2\rho} & 0 & 0 \\ 0 & 0 & 0 & -\frac{15}{\rho} & 0 \\ 0 & 0 & 0 & 0 & -\frac{15}{\rho} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Moreover, since S is a ball its inertia matrix J is a scalar matrix so that the function E defined by (3.15) reduces to

$$E\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} b \times a \\ 0 \end{pmatrix} \qquad a, b \in \mathbb{R}^3.$$

By using the above form of A, B and E we see that the equation for ω_3 in (3.28), (3.29) reduces to

$$\dot{\omega}_3 = -\frac{15}{\rho}\omega_3,$$

which is independent on the control u. This clearly implies that the full system (3.28), (3.29) is not controllable.

The cases in which the family Ψ is a subset of $\mathcal{F}_1$ or of $\mathcal{F}_2$ can be tackled in a similar way, so we skip the corresponding calculations. $\square$

The next result states that if we consider only the system (3.28) not involving the position vector ζ and the rotation R and we assume that S is the unit ball of $\mathbb{R}^3$, we can locally control the velocity field with only three scalar inputs. More precisely the following result holds.

Proposition 5.4. *There exists a open and dense subset $\mathcal{Y}_4$ of $\mathcal{W}^3$ such that the system (3.28) is locally controllable in a neighborhood of the origin, for any family $\{\psi_1, \psi_2, \psi_3\} \subset \mathcal{Y}_4$.*

Proof. It is well-known that the local controllability follows from the controllability of the linearized system which, in our case, reduces to

$$\dot{z}(t) = Az(t) + Bu(t),$$

with the matrix A given by (5.4) and the matrix $B \in M_{6\times 3}(\mathbb{R})$ given by (3.10). The eigenvectors of A have the form

$$\begin{pmatrix} a \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ a \end{pmatrix}, \quad a \in \mathbb{R}^3 \setminus \{0\}.$$

By applying the Hautus test (see for instance Sontag [18, p. 94]) we obtain that (3.28) is controllable if and only if the matrices $B^{(1)}$ and $B^{(2)}$ in (3.10) are invertible. In order to finish the proof it suffices to show that, the set of families $\{\psi_1, \psi_2, \psi_3\} \in \mathcal{W}^3$ such that the 3×3 matrices $B^{(1)}$ and $B^{(2)}$ defined by (3.8) and (3.9) are invertible is open and dense. In order to do that, consider the linear operator $\Lambda_4 : \mathbb{W}' \to [M_3(\mathbb{R})]^2$,

$$\Lambda \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \left(\displaystyle\int_{\partial S} g^{(i)} \cdot \psi_j \, \mathrm{d}\Gamma \right)_{i,j\in\{1,2,3\}} \\ \left(\displaystyle\int_{\partial S} G^{(i)} \cdot \psi_j \, \mathrm{d}\Gamma \right)_{i,j\in\{1,2,3\}} \end{pmatrix},$$

where

$$\mathbb{W}' = \{ (\varphi_i) \in L^2(\partial S; \mathbb{R}^3)^3 \;\; \big| \;\; \varphi_i = 0 \;\text{ outside }\; \Gamma, \quad \varphi_i \cdot n = 0 \text{ on } \partial S, \; i \in \{1,2,3\}\}.$$

A simple calculation shows that for all $(C, D) \in M_3(\mathbb{R})$,

$$\Lambda_4^* \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{k=1}^{3} \left(C_{k,1}(g^{(k)} \times n) \times n + D_{k,1}(G^{(k)} \times n) \times n \right) \\ \displaystyle\sum_{k=1}^{3} \left(C_{k,2}(g^{(k)} \times n) \times n + D_{k,2}(G^{(k)} \times n) \times n \right) \\ \displaystyle\sum_{k=1}^{3} \left(C_{k,3}(g^{(k)} \times n) \times n + D_{k,3}(G^{(k)} \times n) \times n \right) \end{pmatrix} \qquad \text{on } \Gamma.$$

Using Lemma 4.1 2., we deduce that Λ_4^* is one to one. The end of the proof is completely similar to the proof of Theorem 1.1. □

6 Concluding remarks

The results presented here illustrate the controllability properties of a dynamical system modelling the motion of some ocean micro-organisms. Within this model it would be interesting to study in a detailed way the influence of the form of the micro-organism on the controllability properties of the system. We have seen that for a micro-organism having the form of the ball at least six scalar controls seem to be necessary. We conjecture that for less symmetric forms controllability of the full system could be obtained with less than six scalar inputs. One of the major simplifying assumption in the present work is that we neglect the term containing the partial derivative with respect to the time in the equations of the fluid. Further development is therefore required to include this term in the analysis. This question seems difficult since the introduction of this term makes the associated dynamical system genuinely infinite-dimensional. An investigation using this more comprehensive model would provide more insight into the swimming mechanism of micro-organisms.

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References

[1] J. Blake, *A finite model for ciliated micro-organisms*, J. Biomech., 6 (1973), pp. 133–140.

[2] J. R. Blake and S. R. Otto, *Ciliary propulsion, chaotic filtration and a "blinking" Stokeslet*, J. Engrg. Math., 30 (1996), pp. 151–168. The centenary of a paper on slow viscous flow by the physicist H. A. Lorentz.

[3] C. Brennen, *An oscillating-boundary-layer theory for ciliary propulsion*, J. Fluid Mech., 65 (1974), pp. 799–824.

[4] C. Brennen and H. Winet, *Fluid mechanics of propulsion by cilia and flagella*, Ann. Rev. Fluid Mech., 9 (1977), pp. 339–398.

[5] S. Childress, *Mechanics of swimming and flying*, vol. 2 of Cambridge Studies in Mathematical Biology, Cambridge University Press, Cambridge, 1981.

[6] C. Fabre and G. Lebeau, *Prolongement unique des solutions de l'equation de Stokes*, Comm. Partial Differential Equations, 21 (1996), pp. 573–596.

[7] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, vol. 38 of Springer Tracts in Natural Philosophy, Springer-Verlag, New York, 1994. Linearized steady problems.

[8] ——, *On the steady self-propelled motion of a body in a viscous incompressible fluid*, Arch. Ration. Mech. Anal., 148 (1999), pp. 53–88.

[9] ——, *On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications*, in Handbook of mathematical fluid dynamics. Vol. I, North-Holland, Amsterdam, 2002, pp. 653–791.

[10] J. Happel and H. Brenner, *Low Reynolds number hydrodynamics with special applications to particulate media*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1965.

[11] S. Keller and T. Wu, *A porous prolate-spheroidal model for ciliated microorganisms*, J. Fluid Mech., 80 (1977), pp. 259–278.

[12] G. Komatsu, *Analyticity up to the boundary of solutions of nonlinear parabolic equations*, Comm. Pure Appl. Math., 32 (1979), pp. 669–720.

[13] J. Lighthill, *Mathematical biofluiddynamics*, Society for Industrial and Applied Mathematics, Philadelphia, 1975.

[14] C. B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Die Grundlehren der mathematischen Wissenschaften, Band 130, Springer-Verlag New York, Inc., New York, 1966.

[15] J. A. San Martín, V. Starovoitov, and M. Tucsnak, *Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid*, Arch. Ration. Mech. Anal., 161 (2002), pp. 113–147.

[16] D. Serre, *Chute libre d'un solide dans un fluide visqueux incompressible. Existence*, Japan J. Appl. Math., 4 (1987), pp. 99–110.

[17] A. L. Silvestre, *On the slow motion of a self-propelled rigid body in a viscous incompressible fluid*, J. Math. Anal. Appl., 274 (2002), pp. 203–227.

[18] E. D. Sontag, *Mathematical control theory*, vol. 6 of Texts in Applied Mathematics, Springer-Verlag, New York, second ed., 1998. Deterministic finite-dimensional systems.

[19] T. Takahashi and M. Tucsnak, *Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid*, J. Math. Fluid Mech., 6 (2004), pp. 53–77.

[20] G. Taylor, *Analysis of the swimming of microscopic organisms*, Proc. Roy. Soc. London. Ser. A., 209 (1951), pp. 447–461.